Problem Solving Seminar (9/14/09)

(Problems 1-8 are from *The Inquisitive Problem Solver* by Vaderlind, Guy and Larson. Problems 9-12 are from AMC-12 tests.)

1. You have 43 pieces of fencing whose lengths are 1, 2, 3, ..., 43 meters. Can you build a square fence using all of these pieces (without any bending)? (Follow-up question)

2. A sports team has a winning percentage less than 75%. After winning a number of games, the winning percentage is greater than 75%. Was there a time at which the team's winning percentage was exactly 75%?

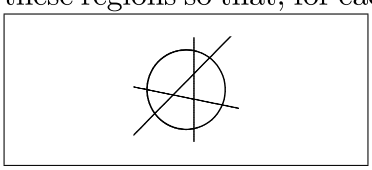
3. Let A, B, C, D, E, F, G, H, I be any nine positive integers and let a, b, c, d, e, f, g, h, i be any permutation (re-arrangement) of those numbers. Can the permutation be chosen so that the product

$$(A-a)(B-b)(C-c)(D-d)(E-e)(F-f)(G-g)(H-h)(I-i)$$

is an odd number?

4. In a group of 20 students, each student is required to buy lunch for 8 other students. Is it always going to happen that there are two students who buy each other's lunch? At what group size does your answer change?

5. Three lines divide a circle into seven regions. Can you arrange the numbers 1 to 7 in these regions so that, for each line, the sums of the numbers on either side are the same?

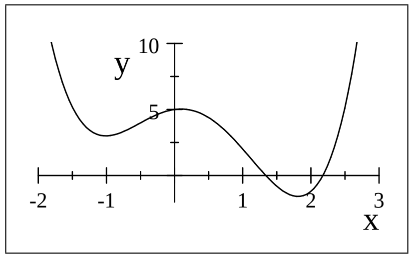


6. In a round-robin tournament, medals are awarded as follows. Player A earns a medal if for each other player B, either A beat B or A beat a player who beat B. Is it possible that no medals will be awarded? Is it possible that everybody will be awarded a medal?

7. Find the last digit of each expression: (a) $4^{4^{4^{4^{4}}}}$ (b) $5^{5^{5^{5^{5}}}}$ (c) $6^{6^{6^{6^{6}}}}$ (d) $7^{7^{7^{7^{7}}}}$ (e) $8^{8^{8^{8^{8}}}}$

8. For what values of n can the set $\{1, 2, 3, ..., n\}$ be partitioned into two or more subsets such that the products of all members of the subsets are equal? (Hint: try $n = 25$.)

9. The graph shows a portion of the graph of $P(x) = x^4 + ax^3 + bx^2 + cx + d$ for constants a, b, c and d. List the following in ascending order: $A = P(-1)$, $B =$ the product of the zeros of P, $C =$ the product of the non-real zeros of P, $D =$ the sum of the coefficients of P, $E =$ the sum of the real zeros of P.



10. Find the product xyz given that $x + \frac{1}{y} = 4$, $y + \frac{1}{z} = 1$ and $z + \frac{1}{x} = \frac{7}{3}$.

11. Find all x such that the mean, median and mode of $\{10, 2, 5, 2, 4, 2, x\}$ form a non-constant arithmetic sequence.

12. A sequence starts with $a_1 = x$, $a_2 = 2000$ and $a_n = a_{n-1}a_{n+1} - 1$. Find all values of x such that 2001 is in the sequence.