

Problem Solving Seminar (9/21/09)

Pigeonhole Principle

1. Let A be any set of 20 distinct integers chosen from the arithmetic progression 1, 4, 7, ..., 100. Prove that there must be two distinct numbers in A whose sum is 104. (From the 1978 Putnam exam.)
2. Five points are situated inside an equilateral triangle whose side has length one unit. Show that two of them may be chosen which are less than one unit apart.
3. Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint subsets whose members have the same sum.
4. Given any $n + 2$ integers, show that there exist two of them whose sum, or else whose difference, is divisible by $2n$.
5. If $a_1 < a_2 < \dots < a_n$, $b_1 > b_2 > \dots > b_n$ and $\{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n\} = \{1, 2, 3, \dots, 2n\}$ show that $\sum_{i=1}^n |a_i - b_i| = n^2$.
6. Show that there is some four-digit combination that occurs infinitely often as the first four digits of the powers of 2.
7. On a 3×7 checkerboard, every square is colored red or blue. Show that in any such coloring there is a rectangle (formed by the lines of the board) whose distinct corner squares are all the same color.
8. Show that for any five integer-valued vectors in \mathbb{R}^2 , at least one pair of them averages to another integer-valued vector.