Problem Solving Seminar (11/3)

(These problems are from Harvey Mudd Putnam seminars)

1. Eight people sit around a table. Each person's age is the average of the two persons' ages on his/her left and right. Show that all their ages are equal.

2. Place the integers 1 through 25 on a 5-by-5 grid in any order with no repetition. Show that there exist two adjacent numbers (horizontally, vertically or diagonally) that differ by at least 6. Can the numbers be placed to avoid adjacent numbers differing by 7 or more?

3. Given 16 positive integers, show that there are two whose difference is a multiple of 15.

4. Show that for any 3 integers, two of them average to another integer.

5. Let $M = a^2 + b^2$ be a sum of two squares. Let N also be a sum of two squares. Show that the product MN is also a sum of two squares.

6. Arrange the integers 1, 2, ..., n consecutively around a circle. Now, remove the number 2 and continue to remove every other number until only one number remains. If $n = 2^p$ what is the last number remaining? If $n = 2^p + 1$, what is the last number remaining?

7. What is the largest integer for which $n + 5$ divides $n^3 + 25$?

8. Without a calculator, determine which is bigger, $\sqrt{19} + \sqrt{99}$ or $\sqrt{20} + \sqrt{98}$.

9. For real numbers, which is bigger, $\left(\sum_{i=1}^{n} x_i\right)^2$ or $n\sum_{i=1}^{n} x_i^n$?

10. You're the last of 100 people in line for 100 seats on an airplane. The first person randomly picks a seat. Everyone else sits in their assigned seat if it's available, or picks a random seat if the seat is already taken. What is the probability that you get to sit in your seat?